

# Radon Transform On Broken Lines and Cone Differentiation

G. Ambartsoumian, M.J. Latifi Jebelli

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## Abstract

In this paper we will provide some related results. First we introduce a generalization of Fundamental Theorem of Calculus that we call **Cone Differentiation** Theorem. Then we use that theorem to derive an explicit inversion formula for **Broken-Ray Radon Transform** and we state a version of that result in the case of weighted rays and also a generalization to higher dimensions. We will also drive a range description for Broken-Ray Radon Transform in two dimensional case.

## 1 Cone Differentiation Theorem

We start with a generalization of fundamental theorem of calculus in  $\mathbb{R}^n$  from partial order perspective. The term cone in **Cone Differentiation** Theorem reflects the concept of positive cone in a partially ordered vector space.

We start our discussion by writing Fundamental Theorem of Calculus in terms of natural order in  $\mathbb{R}$  and then we build the necessary background for our generalization to  $\mathbb{R}^n$ . If  $\leq$  is the natural order on  $\mathbb{R}$ , for an integrable function  $f$  and

$$F(x) = \int_{y \leq x} f(y) dt$$

we have  $F' = f$  almost every where. Note that in this case  $F$  is absolutely continuous.

## 1.1 Partial Order on $\mathbb{R}^n$

Now we look at the concept of partial order in a vector space. A **Partially Ordered Vector Space**  $V$  is a vector space over  $\mathbb{R}$  together with a partial order  $\leq$  such that

1. if  $x \leq y$  then  $x + z \leq y + z$  for all  $z \in V$
2. if  $x \geq 0$  then  $cx \geq 0$  for all  $c \in \mathbb{R}^+$

From the definition we have  $x \leq y \Leftrightarrow 0 \leq y - x$  and hence the order is completely determined by  $V^+ = \{x \in V; x \geq 0\}$  **positive cone of  $V$** . Furthermore, for  $P \subset V$  there is a partial order on  $V$  such that  $P = V^+$  if and only if

$$P \cap (-P) = \{0\}$$

$$P + P \subset P$$

$$c \geq 0 \Rightarrow cP \subset P$$

It is easy to check that for the case of  $V = \mathbb{R}^n$  the positive cone is actually a cone in geometric sense. A topology on  $V$  is said to be *compatible* with order structure if  $V^+$  is closed. We can also identify a partial order structure with  $V^- = -V^+$  **negative cone of  $V$** . For a detailed treatment of these see [3].

In this paper we will consider partial orders in  $\mathbb{R}^n$  corresponding to negative cones  $(\mathbb{R}^n)^-_{\mathcal{B}}$  generated by a set of fixed basis vectors  $\mathcal{B} = \{v_1, \dots, v_n\}$ , i.e.  $(\mathbb{R}^n)^-_{\mathcal{B}} = \{\sum_{i=1}^n c_i v_i; c_i \geq 0\}$ . This partial order is compatible with standard topology of  $\mathbb{R}^n$ .

In the case of  $\mathbb{R}^2$  we will use independent vectors  $u, v$  as a generating set for the negative cone. In this case the boundary of the negative cone is a broken line which is the building block of our construction of inversion formula for broken-line Radon transform.

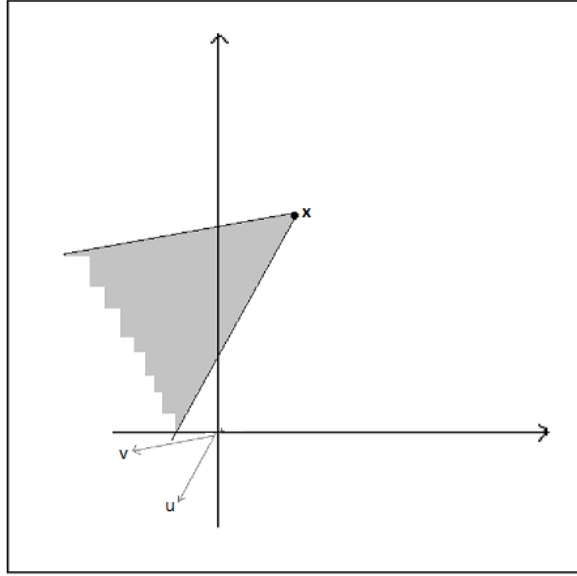
## 1.2 Formulation of the Theorem

Now a natural question is in what sense of differentiation we can generalize Fundamental Theorem of calculus to  $\mathbb{R}^n$ ? One can also relate this problem to local change of the coordinate system, but we will do our specific construction

in a general setting using order structure on  $\mathbb{R}^n$  and its geometric properties. For  $f \in L^1(\mathbb{R}^n)$  we define  $F$  on  $\mathbb{R}^n$  as

$$F(x) = \int_{y \leq x} f(y) d\mu$$

where  $\mu$  is the standard Lebesgue measure on  $\mathbb{R}^n$  and  $y \leq x$  represents the negative cone at  $x$  with respect to partial order on  $\mathbb{R}^n$ . In other words, the integral is taken over the region  $\{y \in \mathbb{R}^n; y \leq x\}$ . Now, by looking at the following figure you can see the negative cone corresponding to  $x$  which appears in the formula:



We will rely on the following theorem (which is a generalization of fundamental theorem of calculus). Let  $B_r(x)$  be the Euclidean ball with radius  $r$  centered at  $x$  then we have the following

**Theorem 1** (Lebesgue differentiation theorem, see [4] and [1]) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be integrable. Then the following is true almost everywhere:

$$f(x) = \lim_{r \rightarrow 0} \frac{1}{B_r(x)} \int_{B_r(x)} f d\mu$$

where  $\mu$  is the Lebesgue measure on  $\mathbb{R}^n$ . The above equality holds everywhere if  $f$  is continuous on  $\mathbb{R}^n$ .

Note that this result holds even if we consider balls coming from another equivalent metric structure on  $\mathbb{R}^n$ , for example if  $B_r(x)$  is a square centered at  $x$ . In fact, the family of balls described above can be replaced with a fairly large family of open sets that **shrink to  $x$  nicely** as explained in [1]. For more comprehensive approach to this concept and its relation to Radon-Nikodym Theorem see chapter 7 of [1].

Now we write Radon-Nikodym Theorem ([1]) :

**Theorem 2** *If  $\mu$  is the Lebesgue measure on  $\mathbb{R}^n$ , and  $\nu$  is a signed measure on  $\mathbb{R}^n$  such that  $\nu \ll \mu$ , i.e.  $\nu$  is absolutely continuous with respect to  $\mu$ , then there is a unique integrable real valued function  $f$  on  $\mathbb{R}^n$  such that for every measurable set  $A$ ,*

$$\nu(A) = \int_A f d\mu$$

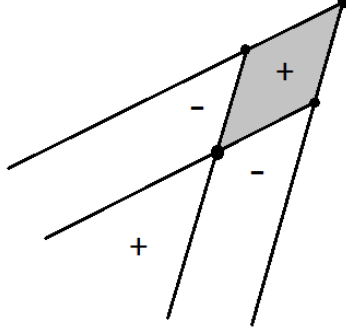
*and  $f$  is called Radon-Nikodym derivative of measure  $\nu$ .*

Furthermore, if  $\nu$  is a measure (not negative) then the function  $f$  will be a nonnegative function.

### 1.3 Cone Differentiation

We start from two dimensions. Let  $f$  be an integrable function on  $\mathbb{R}^2$  (with  $\int |f| < \infty$ ) with respect to Lebesgue measure and define  $F(x) = \int_{y \leq x} f(y) d\mu$  using the partial order made by  $u, v$ . We also define  $V_{t,s}(x)$  as the average of  $f$  over the parallelogram centered at  $x$ , sides of length  $t, s$  and directions  $u, v$ . We can consider these parallelograms as a nicely shrinking family of neighborhoods described in [1]. Note that the area of the parallelogram made with vectors  $tu, sv$  is equal to  $|\det(tu, sv)| = ts |\det(u, v)|$ . With a simple geometric argument and considering  $F$  as an integral over the negative cone we get

$$\begin{aligned} V_{t,s}(x) = \frac{1}{ts |\det(u, v)|} & [F(x + \frac{t}{2}u + \frac{s}{2}v) - F(x - \frac{t}{2}u + \frac{s}{2}v) \\ & - F(x + \frac{t}{2}u - \frac{s}{2}v) + F(x - \frac{t}{2}u - \frac{s}{2}v)] \end{aligned}$$



Likewise, for  $n$  dimensions by a geometric argument and induction over  $n$  we get the following averaging formula for  $f$

$$V_{t_1, \dots, t_n}(x) = \frac{1}{t_1 \dots t_n |\det(v_1, \dots, v_n)|} \sum_{\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}^n} \text{sgn}(\alpha_1 \dots \alpha_n) F(x + \alpha_1 t_1 v_1 + \dots + \alpha_n t_n v_n)$$

Now by averaging over an infinitesimal symmetric neighborhood of  $x$  and applying Theorem 1 we have the following result

**Theorem 3** *Let  $\leq$  be an order in  $\mathbb{R}^n$  made from the positive cone of  $v_1, \dots, v_n$  and for  $f \in L^1(\mathbb{R}^n)$  define*

$$F(x) = \int_{y \leq x} f(y) d\mu.$$

*Then for almost every  $x$  we have*

$$f(x) = \lim_{t \rightarrow 0} V_{t, \dots, t}(x)$$

*where  $V_{t, \dots, t}(x)$  is defined as above letting  $t_1 = \dots = t_n = t$  in the definition of  $V$ .*

Note that this method of recovering  $f$  from  $F$  is of practical significance because it is both efficient and simple. In previous theorem by assuming that  $t_1 = \dots = t_n = t$  we send  $t_1, \dots, t_n$  simultaneously to zero which enables us to use Theorem 1. But we discuss now that in special case when  $f$  is a continuous function we can send them to zero in any order and get the same result.

Now we start the case when  $f$  is continuous on  $\mathbb{R}^2$ ,  $u = e_1$  and  $v = e_2$ . For calculating the average of a continuous function  $f$  over a rectangle with center  $(x, y)$  and  $2t$  and  $2s$  respectively as width and height, we have:

$$V_{2t, 2s}(x_0, y_0) = \frac{1}{4ts} \int_{x_0-t}^{x_0+t} \int_{y_0-s}^{y_0+s} f(x, y) dx dy$$

Now both of the integrals are over a compact interval and the function is continuous. By taking the limit as  $s \rightarrow 0$ , pushing the limit inside the integral and averaging  $f$  we get

$$\frac{1}{2t} \int_{x_0-t}^{x_0+t} f(x, y_0) dx$$

and then if  $t \rightarrow 0$  we will get the value of  $f$  at  $(x, y)$ :

$$\lim_{t \rightarrow 0} \lim_{s \rightarrow 0} V_{t,s}(x_0, y_0) = f(x_0, y_0)$$

Based on this observation for continuous functions, instead of sending width and height of the averaging region simultaneously to zero we can send these two quantities to zero in order and still get the same result at the point. Note that because of the fact that we can push the limit inside the integral for continuous  $f$  the order of taking limits makes no difference. In the case of arbitrary  $u$  and  $v$  the same argument will work after applying a change of variable. One can see that this is also true even if we increase the dimension. Now, we can prove our main result

**Theorem 4** *Let  $\leq$  be as in Theorem 3 and  $F(x) = \int_{t \leq x} f(t) d\mu$  with  $f$  continuous, then we have*

$$f(x) = \frac{1}{|\det(v_1, \dots, v_n)|} \frac{\partial}{\partial v_1} \dots \frac{\partial}{\partial v_n} F(x).$$

Proof. Using the fact that we can apply the limits in order, by Theorem 3 we have

$$f(x) = \lim_{t_n \rightarrow 0} \dots \lim_{t_1 \rightarrow 0} V_{t_1, \dots, t_n}(x)$$

multiplying both sides of this by  $|\det(v_1, \dots, v_n)|$  and using the definition of  $V_{t_1, \dots, t_n}(x)$  we need to show

$$\frac{\partial}{\partial v_n} \dots \frac{\partial}{\partial v_1} F(x) = \lim_{t_n \rightarrow 0} \dots \lim_{t_1 \rightarrow 0} \frac{1}{t_1 \dots t_n} \sum_{\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}^n} \text{sgn}(\alpha_1 \dots \alpha_n) F(x + \alpha_1 t_1 v_1 + \dots + \alpha_n t_n v_n)$$

By induction on dimension for  $n = 1$  from definition of derivative and the fact that  $f$  is continuous we have

$$\frac{\partial F(x)}{\partial v_1} = \lim_{t_1 \rightarrow 0} \frac{F(x + t_1 v_1) - F(x)}{t_1} = \lim_{t_1 \rightarrow 0} \frac{F(x + \frac{t_1}{2} v_1) - F(x - \frac{t_1}{2} v_1)}{t_1}$$

Now assume the equality is true for dimension  $n - 1$  and apply  $\partial/\partial v_n$  to both sides. Then on the right hand side we have

$$\begin{aligned} \frac{\partial}{\partial v_n} \left( \frac{\partial}{\partial v_{n-1}} \cdots \frac{\partial}{\partial v_1} F(x) \right) &= \lim_{t_{n-1} \rightarrow 0} \cdots \lim_{t_1 \rightarrow 0} \frac{1}{t_1 \cdots t_{n-1}} \\ &\quad \sum_{\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}^n} \text{sgn}(\alpha_1 \cdots \alpha_{n-1}) \frac{\partial}{\partial v_n} (F(x + \alpha_1 t_1 v_1 + \cdots + \alpha_{n-1} t_{n-1} v_{n-1})) \end{aligned}$$

hence using

$$\begin{aligned} \frac{\partial F}{\partial v_n}(x + \alpha_1 t_1 v_1 + \cdots + \alpha_{n-1} t_{n-1} v_{n-1}) &= \lim_{t_n \rightarrow 0} \frac{1}{t_n} [F(x + \alpha_1 t_1 v_1 + \cdots + \alpha_{n-1} t_{n-1} v_{n-1} + \frac{t_n}{2} v_n) \\ &\quad - F(x + \alpha_1 t_1 v_1 + \cdots + \alpha_{n-1} t_{n-1} v_{n-1} - \frac{t_n}{2} v_n)] \end{aligned}$$

we get the result •

In two dimensional case we can rewrite the theorem in the following way:

$$f(x) = \frac{1}{|\det(u, v)|} \frac{\partial}{\partial u} \frac{\partial}{\partial v} F(x)$$

Now we give an answer to the question that what is the necessary and sufficient condition for a function  $F$  to be a cone integral of another function  $f$  with respect to a given order structure in  $\mathbb{R}^n$ , namely existence of  $f$  such that  $F(x) = \int_{y \leq x} f(y) d\mu$ .

We will apply the Radon Nikodym Theorem to get the desired description of  $F$ . For a given  $F$ , we construct a corresponding signed measure  $\nu$  that implies existence of  $f$ . For  $x \in \mathbb{R}^n, c_i \in \mathbb{R}^+$ , let  $P(x, c_1, \dots, c_n)$  be a  $n$ -dimensional half-open parallelogram defined by

$$P(x, c_1, \dots, c_n) = \{x + \sum_{i=1}^n t_i v_i \in \mathbb{R}^n; -c_i \leq t_i < c_i\}$$

These parallelograms are the analogs of intervals in one dimension. We define a pre-measure  $\nu_0$  on the ring of subsets generated by these parallelograms using

$$\nu_0(P(x, c_1, \dots, c_n)) = \sum_{\alpha \in \{-1, 1\}^n} \text{sgn}(\alpha_1 \cdots \alpha_n) F(x + \alpha_1 c_1 v_1 + \cdots + \alpha_n c_n v_n)$$

Using a natural relation to absolute continuity on the real line we define absolute continuity of  $F$  as follows:

**Definition 1** *Let  $F$  be a function on  $\mathbb{R}^n$  with a given order structure of  $\mathbb{R}^n$ . Then we say  $F$  is absolutely continuous if and only if for any  $\epsilon > 0$  there exist  $\delta > 0$  such that for any finite collection of parallelograms  $\{P_i\}_{i=1}^n$ ,  $\sum_{i=1}^n \mu(P_i) < \delta$  implies  $\sum_{i=1}^n \nu_0(P_i) < \epsilon$ .*

By applying Caratheodory extension Theorem, we can extend  $\nu_0$  to a measure  $\nu$  on  $\mathbb{R}^n$  (domain of  $\nu$  is the  $\sigma$ -algebra of Lebesgue measurable sets).

Now let  $\alpha = \frac{v_1 + \dots + v_n}{\|v_1 + \dots + v_n\|}$  be the direction of negative cone. Then, we observe that for  $F(x) = \int_{t \leq x} f(t) d\mu$  if we send  $x$  in opposite direction of  $\alpha$ , starting from zero, we get the integral of  $f$  over  $\mathbb{R}^n$  as a limit. In other words, we have

$$\lim_{t \rightarrow -\infty} F(t\alpha) = \int_{\mathbb{R}^n} f d\mu$$

Note that if  $f \geq 0$  then  $F$  will be increasing with respect to order structure and in other direction if  $F$  is increasing with respect to a given partial order then we can conclude that the constructed signed measure  $\nu$  will be nonnegative and hence  $\nu$  will be simply a measure. Following is a consequence of Radon Nikodym Theorem:

**Theorem 5** *For arbitrary increasing function  $F$  on  $\mathbb{R}^n$  with  $\lim_{t \rightarrow -\infty} F(t\alpha) < \infty$ ,  $F$  is absolutely continuous if and only if there exist a nonnegative real valued function  $f$  on  $\mathbb{R}^n$  such that*

$$F(x) = \int_{y \leq x} f d\mu$$

Proof. Assume for nonnegative function  $f$  we have

$$F(x) = \int_{y \leq x} f d\mu$$

then by  $\lim_{t \rightarrow -\infty} F(t\alpha) < \infty$  we know that  $f$  is integrable and hence for any  $\epsilon > 0$  there exist a  $\delta > 0$  such that  $\mu(E) < \delta$  implies

$$\int_E f d\mu < \epsilon$$

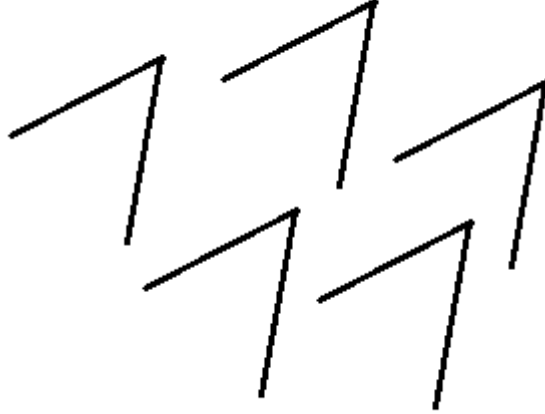
therefore  $F$  is absolutely continuous. Other direction is an implication of Radon Nikodym theorem •

We will use this theorem later to provide a range description for Broken-Ray Radon Transform.



## 2 Broken-Ray Radon Transform

By Radon Transform on broken lines we mean integrating over lines of the form



when all broken lines have the same direction. We define generators of the negative cone  $u, v$  to be the direction of two broken parts. Hence for any particular broken-line problem we have a partial order structure on  $\mathbb{R}^n$ .

### 2.1 Perpendicular Broken Lines

To make the idea clear, first we consider the special case of right angle broken lines. For  $I = [0, 1]$ , we study the operator on  $L(I^2)$  defined by:

$$(Tf)(x, y) = \int_0^x f(t, y)dt + \int_0^y f(x, t)dt \quad (1)$$

We can see this as a Radon Transform that integrates values of a function along all possible broken lines consisting of two perpendicular segments.

We have the following lemma to explain the special case considering standard partial order on  $\mathbb{R}^2$  by letting  $u = e_1, v = e_2$ . We will proof a more general result using measure theory techniques in next section.

**Lemma 1** *For the operator defined as above, let*

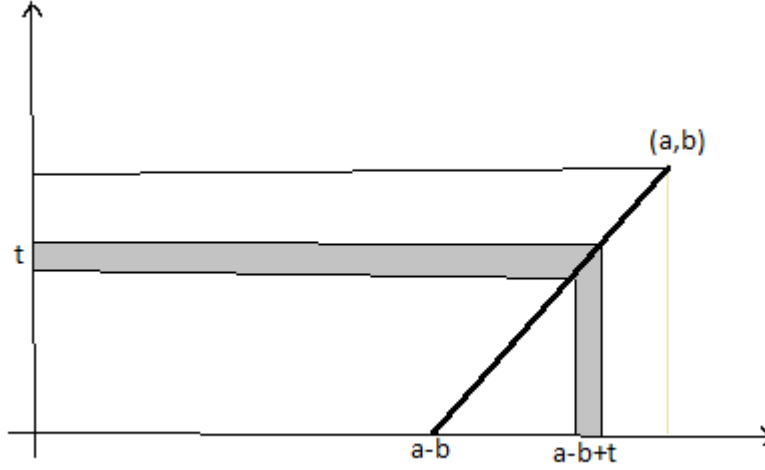
$$F(x, y) = \int_0^y (Tf)(x - y + t, t)dt.$$

Then  $F$  is the integral of  $f$  over the negative cone at  $(x, y)$ . Also for continuous function  $f$  we have

$$f(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} F(x, y)$$

Proof. We only need to show the first part, the rest of the statement is a consequence of Cone Differentiation Theorem in the previews section.

Now, let  $g = Tf$ , then  $g(x, y)$  represents the area under  $f$  on the broken line with vertex at  $(x, y)$ . We integrate infinitesimal sections described in the following figure to establish a formula for the volume  $V(a, b)$  under  $f$  over the region  $[0, a] \times [0, b]$



The length of the broken line at a given time  $0 \leq t \leq b$  is  $a - b + 2t$ , hence the average value of function on the corresponding line segment is  $\frac{g(a-b+t, t)}{a-b+2t}$ . Multiply this with the volume of the infinitesimal segment we get:

$$F(a, b) = \int_0^b \frac{g(a-b+t, t)}{a-b+2t} (a-b+2t) dt = \int_0^b g(a-b+t, t) dt$$

and this completes the proof •

## 2.2 General Broken Lines

Now we prove the main result for broken lines. Assume that  $L(x, y)$  is the unique cone at  $(x, y)$  with direction  $\alpha$ , where  $\alpha = (\alpha_x, \alpha_y)$  is a unit vector parallel to  $\frac{u+v}{2}$ . Also, let  $\beta$  to be the angle between  $u$  and  $\alpha$ .

**Theorem 6** Let  $T$  be the operator on  $L^1(\mathbb{R}^2)$  defined by:

$$(Tf)(x, y) = \int_{L(x, y)} f dL$$

and let

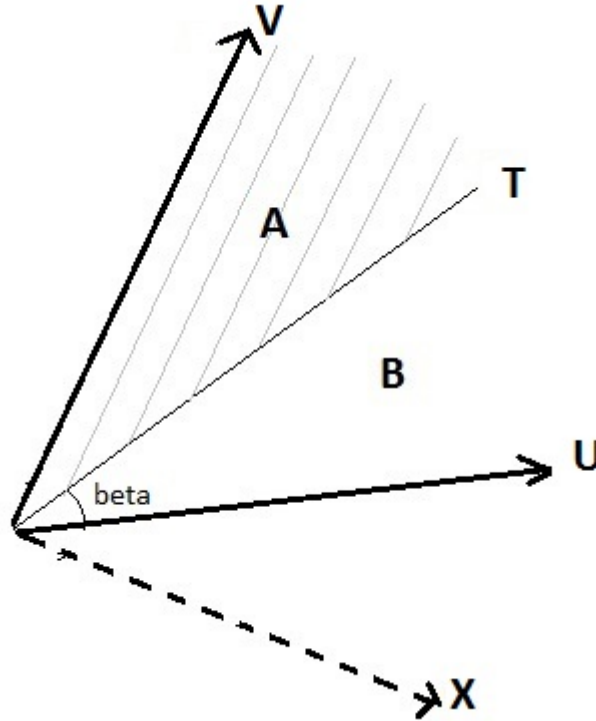
$$F(x, y) = \int_0^\infty (Tf)(x + t\alpha_x, y + t\alpha_y) \sin\beta dt.$$

Then  $F$  is the integral of  $f$  over the negative cone at  $(x, y)$ . Also define  $V_{t,t}(x, y)$  as in the previous section then we have  $f(x, y) = \lim_{t \rightarrow 0} V_{t,t}(x, y)$  almost everywhere and specially if  $f$  is continuous we have

$$f(x, y) = \frac{1}{|\det(u, v)|} \frac{\partial}{\partial u} \frac{\partial}{\partial v} F(x, y).$$

Proof. Once again we only need to show the first part, rest of the statement is a consequence of Cone Differentiation Theorem.

Let  $g = Tf$  and  $\gamma(t)$  be the parametric equation of the ray starting at  $(x, y)$  and moving in direction of  $\alpha$ , namely  $\gamma(t) = (x, y) + t\alpha$ . This curve divides the region enclosed by broken plane into two region we call them  $A, B$ . Now we choose a specific coordinate system to see  $A$  in terms of it's sections:

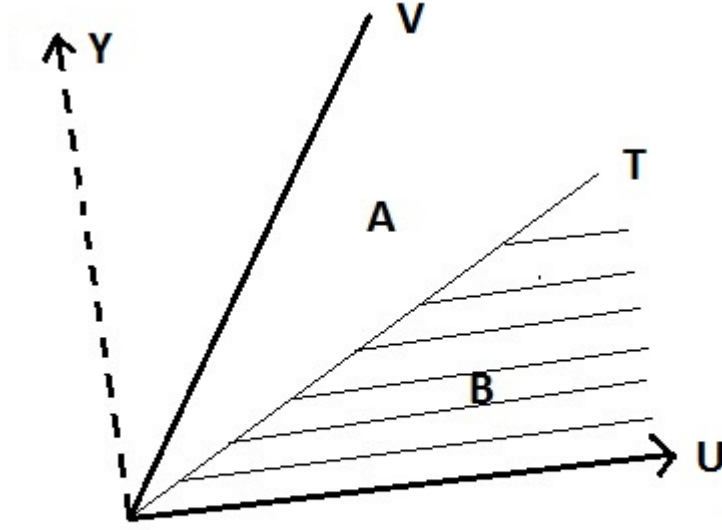


in  $V, X$  coordinate and using  $A_x$  as  $x$ -section of  $A$ , using Fubini Theorem we have:

$$\int_A f d\mu = \int_0^\infty \int_{A_x} f d\mu_V d\mu_X = \int_0^\infty \int_{A_t} f d\mu_V \sin(\beta) d\mu_T$$

where  $\mu_V, \mu_X, \mu_T$  are arc-length measure on  $v, x, t$  respectively, and we have  $d\mu_X = \sin(\beta) d\mu_T$ . Similarly for  $B$  we choose another coordinate system:

$$\int_B f d\mu = \int_0^\infty \int_{B_y} f d\mu_U d\mu_Y = \int_0^\infty \int_{B_t} f d\mu_U \sin(\beta) d\mu_T$$



Adding these two we get the formula

$$\begin{aligned} F(x, y) &= \int_{A \cup B} f d\mu = \int_0^\infty \left( \int_{A_t} f d\mu_U + \int_{B_t} f d\mu_V \right) \sin(\beta) d\mu_T \\ &= \int_0^\infty g(x + t\alpha_x, y + t\alpha_y) \sin(\beta) dt \end{aligned}$$

•

To compare this version with the perpendicular case let  $\beta = \pi/4$ , we have:

$$\int_0^{b\sqrt{2}} g(a - t\sqrt{(2)/2}, b - t\sqrt{2}/2)(\sqrt{2}/2) dt = \int_0^b g(a - b + s, s) ds \quad (2)$$

where  $s = b - t\sqrt{2}/2$ .

In the following examples we applied theorem for perpendicular broken lines:

**Example 1** Let  $f(x, y) = 3x^2 + 3y^2$  then  $Tf(x, y) = g(x, y) = (x + y)^3$ , using the inversion formula we get

$$F(x, y) = \int_0^y (x - y + 2t)^3 dt = xy(x^2 + y^2)$$

and by applying the derivative we get our function back.

**Example 2** Let  $f(x, y) = ye^x$  then  $g(x, y) = ye^x + y^2e^x/2 - y$  and

$$F(x, y) = \int_0^y te^{x-y+t} + \frac{t^2}{2}e^{x-y+t} - tdt = \frac{1}{2}(e^x - 1)y^2$$

and we get  $ye^x$  back by taking derivative.

## 2.3 Compare to known results

Now we want to compare the inversion formula by that of Gouia-Zarrad, and Ambartsoumian [2]:

**Theorem 7** Consider a function  $f(x, y) \in C_c^\infty(D)$ , where  $D = \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq x_{max}, 0 \leq y \leq y_{max}\}$ , defining Radon transform at  $(x, y)$  by

$$g(x, y) = \int_{C(x, y)} f(x, y) ds$$

where  $C(x, y)$  is upward cone starting at  $(x, y)$ , then we have inversion formula:

$$f(x, y) = \frac{-\cos\beta}{2} \left[ \frac{\partial}{\partial y} g(x, y) + \tan^2(\beta) \int_y^{y_{max}} \frac{\partial^2}{\partial x^2} g(x, t) dt \right].$$

Note that in this result we have upward cones which means  $\alpha = (0, 1)$  in our setup. Using the fact that for a smooth function vanishing at infinity we have  $f(y) = \int_y^\infty -f'(t)dt$ , We can rewrite this inversion formula as

$$f(x, y) = \frac{\sin\beta}{2} \int_y^{y_{max}} \left( \cot(\beta) \frac{\partial^2}{\partial y^2} - \tan(\beta) \frac{\partial^2}{\partial x^2} \right) g(x, t) dt$$

Now we apply our inversion formula to compare. First,  $\beta$  is half of the opening angle between  $u, v$  and we rewrite  $\det(u, v)$  as  $\sin(2\beta)$ . Now we need a coordinate transformation to change partials to  $u, v$  coordinate system where  $u$  and  $v$  are unit vectors that generate our upward cone. By a simple argument we have

$$\begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \begin{bmatrix} \sin\beta & \cos\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}$$

Therefor we have:

$$\frac{\partial^2}{\partial v \partial u} = \cos^2(\beta) \frac{\partial^2}{\partial y^2} - \sin^2(\beta) \frac{\partial^2}{\partial x^2}$$

Applying this to our volume formula  $V(x, y) = \sin(\beta) \int_0^\infty g(x, y + t) dt$  and passing differentiation through integral we get

$$f(x, y) = \frac{1}{\sin 2\beta} \frac{\partial^2 V(x, y)}{\partial v \partial u} = \frac{\sin\beta}{\sin 2\beta} \int_0^\infty \left( \cos^2(\beta) \frac{\partial^2}{\partial y^2} - \sin^2(\beta) \frac{\partial^2}{\partial x^2} \right) g(x, y+t) dt$$

And after simplification we get an identical result as in [2].

## 2.4 Range Description

**Lemma 2** For integrable function  $f$  on  $\mathbb{R}^2$  and  $(x, y) \in \mathbb{R}^2$  we have

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{(x,y) \leq z \leq (x,y) + \epsilon \alpha} f d\mu = \sin \beta \int_{L(x,y)} f dL$$

almost everywhere, where  $L(x, y)$  is the broken line at  $(x, y)$ .

Proof. Let  $h(t) = \int_{L((x,y)+t\alpha)} f dL$  then using the same argument as in the proof of Theorem 6 we get

$$\int_{(x,y) \leq z \leq (x,y) + \epsilon \alpha} f d\mu = \int_0^\epsilon h(t) \sin \beta dt = \sin \beta \int_0^\epsilon h(t) dt$$

taking limit from both sides we get desired result almost everywhere •

We have the following range description result using Theorem 5

**Theorem 8** (Range Description) A nonnegative function  $g$  on  $\mathbb{R}^2$  with  $\int_{-\infty}^{+\infty} g(t\alpha_x, t\alpha_y) dt < \infty$  is the image of some nonnegative function  $f$  under Broken Ray transform if and only if function  $F$  defined by

$$F(x, y) = \int_0^\infty g(x + t\alpha_x, y + t\alpha_y) \sin \beta dt$$

is absolutely continuous in the sense of Definition 1

Proof. Let  $g$  be the image of nonnegative function  $f \in L^1(\mathbb{R}^2)$  under Broken ray transform, then we have

$$F(x, y) = \sin\beta \int_0^\infty \int_{L(x+t\alpha_x, y+t\alpha_y)} f dL dt = \int_{z \leq (x, y)} f d\mu$$

and hence by theorem 5,  $F$  is absolutely continuous.

For the other direction, assume that  $F$  is absolutely continuous then  $f$ , if exists, is integrable because

$$\int_{\mathbb{R}^2} f d\mu = \lim_{t \rightarrow -\infty} F(t\alpha_x, t\alpha_y) = \int_{-\infty}^{+\infty} g(t\alpha_x, t\alpha_y) \sin\beta dt < \infty$$

and by Theorem 5 there exists a function  $f$  such that

$$F(x, y) = \int_0^\infty g(x + t\alpha_x, y + t\alpha_y) \sin\beta dt = \int_{z \leq (x, y)} f d\mu$$

by previews lemma we have

$$\lim_{\epsilon \rightarrow 0} \frac{F(x, y) - F(x + \epsilon\alpha_x, y + \epsilon\alpha_y)}{\epsilon} = \sin\beta \int_{L(x, y)} f d\mu$$

but on the left hand side we get the following almost every where

$$\lim_{\epsilon \rightarrow 0} \int_0^\epsilon g(x + t\alpha_x, y + t\alpha_y) \sin\beta dt = \sin\beta g(x, y)$$

which implies  $g(x, y) = \int_{L(x, y)} f d\mu$ . •

## 2.5 Weighted Broken Line Transform

For  $(x, y) \in \mathbb{R}^2$  we define  $X_u(x, y) = \{(x, y) + tu; t \geq 0\}$  as a ray starting from  $(x, y)$  in the direction of  $u$ . One can see any broken line as a disjoint union  $X_u(x, y) \cup X_v(x, y)$  and hence Broken Line transform at any point is the sum of integrals over two corresponding rays. Now, as a generalization, for constants  $c_1, c_2$  we define Weighted Broken Line transform

$$(Tf)(x) = c_1 \int_{X_v(x, y)} f dL + c_2 \int_{X_u(x, y)} f dL$$

where  $dL$  as in previews discussion is the standard Lebesgue measure on line. It is clear from this definition that for  $c_1 = c_2 = 1$  we get the Broken Line transform.

In order to have an inversion formula, we have a similar result on finding  $F(x, y)$  from  $g = Tf$  and the rest will be just applying the Cone Differentiation theorem as we did in preview section.

**Theorem 9** *For Weighted Broken Ray transform defined on  $L^1(\mathbb{R}^2)$  by  $(Tf)(x, y) = c_1 \int_{X_v(x, y)} fdL + c_2 \int_{X_u(x, y)} fdL$  we have*

$$F(x, y) = \frac{\sin(\beta_1)}{c_1} \int_0^\infty (Tf)(x + \alpha_x t, y + \alpha_y t) dt$$

where  $\alpha$  is a vector satisfying

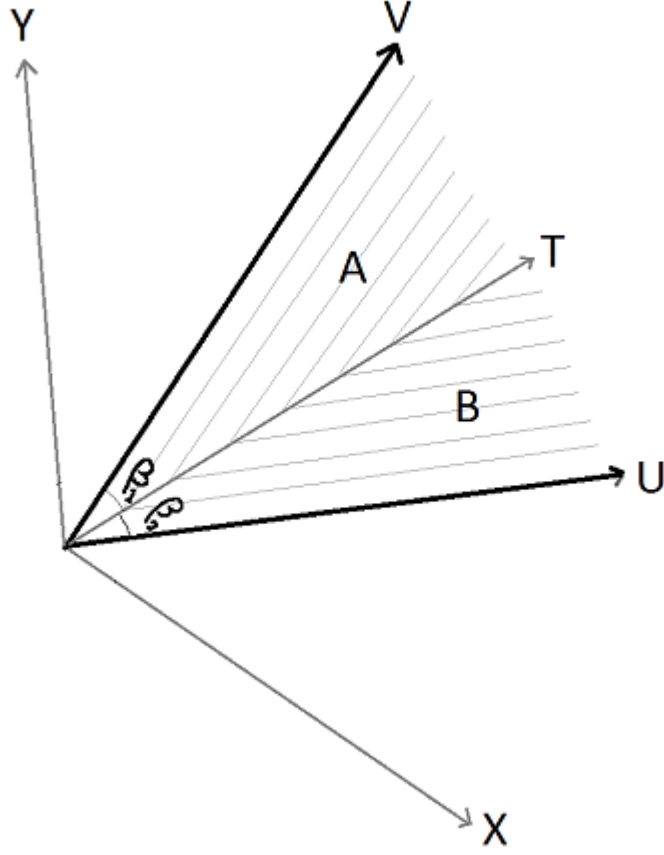
$$\text{angle}(\alpha, v) = \beta_1 \leq \pi$$

$$\text{angle}(\alpha, u) = \beta_2 \leq \pi$$

$$\frac{\sin(\beta)}{\sin(\beta_2)} = \frac{c_1}{c_2}$$

Proof. We follow the same steps as in the proof of non weighted version but this time we divide integration region using the  $\alpha$  defined in the statement of the theorem.





We have

$$\int_A f d\mu = \int_0^\infty \int_{A_x} f d\mu_V d\mu_X = \int_0^\infty \int_{A_t} f d\mu_V \sin(\beta_1) d\mu_T$$

$$\int_B f d\mu = \int_0^\infty \int_{B_y} f d\mu_U d\mu_Y = \int_0^\infty \int_{B_t} f d\mu_U \sin(\beta_2) d\mu_T$$

Adding these we get

$$F(x, y) = \int_{A \cup B} f d\mu = \int_0^\infty \left( \sin(\beta_1) \int_{A_t} f d\mu_V + \sin(\beta_2) \int_{B_t} f d\mu_U \right) d\mu_T$$

using  $c_1/c_2 = \sin(\beta_1)/\sin(\beta_2)$  we have

$$\frac{\sin(\beta_1)}{c_1} \int_0^\infty \left( c_1 \int_{A_t} f d\mu_V + c_2 \int_{B_t} f d\mu_U \right) d\mu_T$$

$$= \frac{\sin(\beta_1)}{c_1} \int_0^\infty (Tf)(x + \alpha_x t, y + \alpha_y t) dt$$

and this finishes the proof •

## 2.6 Conic Radon Transform in Higher Dimensions

We will consider a generalization of our result to higher dimension. For any  $x \in \mathbb{R}^n$ , we define  $(Tf)(x)$  to be integral over the boundary of polyhedral cone  $C$  generated by unit basis vectors  $u_1, \dots, u_n$  starting from  $x$ .

$$(Tf)(x) = \int_{\partial C} f dS$$

Where  $dS$  is the standard  $n - 1$  dimensional Lebesgue measure on  $\partial C$ .

We assume one more condition to make sure that we can break the polyhedral cones to  $n$  congruent subcones. We want  $\|u_i - u_j\|$  to be a fix constant for any choice of  $i$  and  $j$ . In other words,  $w = \frac{u_1 + \dots + u_n}{\|u_1 + \dots + u_n\|}$  is a unit vector with equal angle to every face of polyhedral cone. Let  $X_i = \text{span}\langle u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \rangle$  be the hyperplane containing a face of polyhedral cone and define  $y_i$  to be a unit vector in  $X_i^\perp$ , we have

$$\langle w, y_i \rangle = \langle w, y_j \rangle \quad (i, j = 1, \dots, n)$$

this inner product equality will play a crucial rule in our proof. We have the following inversion Theorem.

**Theorem 10** *For  $T, w, y_j$  defined as above, then*

$$F(x) = \int_0^\infty (Tf)(x + wt) \langle w, y_1 \rangle dt$$

*is integral of  $f$  over the cone generated by  $u_1, \dots, u_n$  starting from  $x$ . Hence, by applying Theorem 3 or 4 we can recover  $f$  from  $F$ .*

Proof. We apply a similar approach as in two dimensional case by replacing regions  $A, B$  in that proof with  $\{A_i\}_{i=1}^n$ . Let  $C$  be the cone at  $x$ , then  $C = \cup_{i=1}^n A_i$  where  $A_i$  is the cone made by  $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$  and  $w$ . In other words we break  $C$  into symmetric disjoint components using vector  $w$ .

Now, to integrate  $f$  over  $C$  we write

$$\int_C f d\mu = \sum_{i=1}^n \int_{A_i} f d\mu$$

Introducing a new axis along  $y_i$  we can write

$$\int_{A_i} f d\mu = \int_0^\infty \int_{E_{y_i}} f d\mu_X d\mu_{Y_i}$$

where  $\mu_X$  is the Lebesgue measure on hyperplane  $X_i$ , orthogonal to  $y_i$ , and  $\mu_{y_i}$  is Lebesgue measure along  $Y_i$  axis.

Now we do the same trick as before. We add another axis  $T$  along vector  $w$  and rewrite above integral using Lebesgue measure on  $T$ . Note that  $\langle w, y_i \rangle$  is the cosine of angle between  $w$  and  $y_i$  and using  $d\mu_{Y_i} = \langle w, y_i \rangle d\mu_T$

$$\int_{A_i} f d\mu = \int_0^\infty \int_{E_t} f d\mu_X \langle w, y_i \rangle d\mu_T$$

but, using the fact that  $\langle w, y_i \rangle = \langle w, y_j \rangle$  we have

$$\begin{aligned} \int_A f d\mu &= \sum_{i=1}^n \int_{A_i} f d\mu = \sum_{i=1}^n \int_0^\infty \int_{E_t} f d\mu_X \langle w, y_i \rangle d\mu_T \\ &= \int_0^\infty \left( \sum_{i=1}^n \int_{E_t} f d\mu_X \right) \langle w, y_1 \rangle d\mu_T \\ &= \int_0^\infty [(Tf)(x + tw)] \langle w, y_1 \rangle d\mu_T \end{aligned}$$

and this finish the proof•

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